

0 Nonnegative integers

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Leopold Kronecker (1823–1891) famously declared that “God made the whole numbers, all else is the work of man.” This statement may refer either to the set of nonnegative integers or to the larger set of all integers. This chapter focuses exclusively on the smaller set. Although these numbers were familiar to ancient civilizations, a comprehensive understanding of them emerged only after more than two millennia of mathematical development.

0.0 Principle of Induction

What are nonnegative integers $\mathbb{N} := \{0, 1, 2, \dots\}$? To understand this fundamental set, we examine an axiomatic characterization.

0.0.0 Definition (Peano axioms). The *nonnegative integers* are a set $\mathbb{N}$ containing a distinguished element 0, called *zero*, together with a successor function $S: \mathbb{N} \rightarrow \mathbb{N}$ satisfying the following conditions.

(C0) Zero is not the successor of any nonnegative integer:

$$S(n) \neq 0 \text{ for all } n \in \mathbb{N}.$$

(C1) The successor function $S: \mathbb{N} \rightarrow \mathbb{N}$ is injective: for all $m, n \in \mathbb{N}$, the equation $S(m) = S(n)$ implies that $m = n$.

(C2) For every subset $\mathcal{X} \subseteq \mathbb{N}$ such that $0 \in \mathcal{X}$ and $S(x) \in \mathcal{X}$ for all $x \in \mathcal{X}$, we have $\mathcal{X} = \mathbb{N}$.

0.0.1 Remark. By design, the successor function $S: \mathbb{N} \rightarrow \mathbb{N}$ captures the essence of counting. Each $n \in \mathbb{N}$ has an assigned successor $S(n)$: $1 := S(0)$, $2 := S(1)$, $3 := S(2)$, $\dots$. Condition (C0) asserts that counting starts at 0, while (C1) ensures that no number is repeated in the counting process. More significantly, Condition (C2) provides a set-theoretic formulation of the principle of induction.

A *model* of the Peano axioms is a triple $(\mathbb{N}, 0, S)$, where $\mathbb{N}$ is a set, $0 \in \mathbb{N}$, and $S: \mathbb{N} \rightarrow \mathbb{N}$ satisfies conditions (C0)–(C2). The next result indicates that the Peano axioms can be taken as a definition of the nonnegative integers.

0.0.2 Theorem. *Any two models of the Peano axioms are isomorphic: for any models $(\mathbb{N}, 0, S)$ and $(\mathbb{N}', 0', S')$, there exists a unique bijective map $\varphi: \mathbb{N} \rightarrow \mathbb{N}'$ such that $\varphi(0) = 0'$ and $S' \circ \varphi = \varphi \circ S$.*

Outline of proof. To prove existence, the map $\varphi: \mathbb{N} \rightarrow \mathbb{N}'$ is defined recursively by $\varphi(0) = 0'$ and $\varphi(S(n)) = S'(\varphi(n))$ for all $n \in \mathbb{N}$. To prove uniqueness, consider another map $\varphi': \mathbb{N} \rightarrow \mathbb{N}'$ with the stated properties. One shows, by induction, that $\varphi(n) = \varphi'(n)$ for all $n \in \mathbb{N}$. By interchanging $\mathbb{N}$ and $\mathbb{N}'$, one similarly obtains a map $\psi: \mathbb{N}' \rightarrow \mathbb{N}$ such that $\psi(0') = 0$ and $S \circ \psi = \psi \circ S'$. One shows, again by induction, that $\psi \circ \varphi = \text{id}_{\mathbb{N}}$ and $\varphi \circ \psi = \text{id}_{\mathbb{N}'}$. Thus, the map φ is a bijection. $\square$

Condition (C2) may be more familiar in an alternative form. A *property* of the set $\mathbb{N}$ is simply a function $P: \mathbb{N} \rightarrow \{\text{true}, \text{false}\}$. We say

The dictum “Die ganzen Zahlen hat der liebe Gott gemacht, alles andere ist Menschenwerk” was uttered by Kronecker in Berlin in 1886, as recounted in the article: H. Weber, *Leopold Kronecker*, *Mathematische Annalen* **43** (1893) 1–25.

The Italian mathematician **Giuseppe Peano** (1858–1932) introduced these axioms in his treatise: G. Peano, *Arithmetices principia, nova methodo exposita*, Bocca, Torino, 1889.

Although the original statement used 1 instead of 0 as the first number, the set $\mathbb{N}$ must include zero to guarantee that it contains the cardinality of every finite set. Peano himself corrects this oversight in his book: G. Peano, *Formulario Mathematico*, Bocca, Torino, 1908.

In set theory, the triple $(\mathbb{N}, 0, S)$ is constructed recursively as

$$\begin{aligned} 0 &:= \{\} = \emptyset \\ 1 &:= \{0\} = 0 \cup \{0\} = \{\emptyset\} \\ 2 &:= \{0, 1\} = 1 \cup \{1\} = \{\emptyset, \{\emptyset\}\} \\ 3 &:= \{0, 1, 2\} = 2 \cup \{2\} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} \\ &\vdots \end{aligned}$$

and the successor function $S: \mathbb{N} \rightarrow \mathbb{N}$ is defined by $S(n) := n \cup \{n\}$.

This theorem should be regarded as a modern interpretation of the “simply infinite systems” appearing in §6 of R. Dedekind, *Was sind und was sollen die Zahlen?*, 4th edition, Braunschweig, 1918. [English translation](#) by W. Beman.

that ‘the property $P(n)$ holds for some $n \in \mathbb{N}$ ’ if $P(n) = \text{true}$. The common form of induction is as follows.

0.0.3 Theorem. To show that a property $P(n)$ holds for all $n \in \mathbb{N}$, it suffices to prove that

(Base case) $P(0)$ holds, and

(Induction step) for all $n \in \mathbb{N}$, the assumption that $P(n)$ holds implies that the property $P(S(n)) = P(n + 1)$ holds.

Proof. Each property P satisfied by (some or all of) the elements in $\mathbb{N}$ corresponds to a subset of $\mathbb{N}$, namely $\mathcal{X} := \{m \in \mathbb{N} \mid P(m) \text{ holds}\}$. Condition (C2) establishes that $\mathcal{X} = \mathbb{N}$. $\square$

The next two problems typify the basic use of induction. Notice that the base case may not be 0.

0.0.4 Problem. For every positive integer n , verify that

$$\sum_{j=0}^{n-1} (2j + 1) = n^2.$$

Inductive solution. When $n = 1$, we have $2(0) + 1 = (1)^2$, so the base case holds. Assuming that $\sum_{j=0}^{n-2} (2j + 1) = (n - 1)^2$, the induction step is

$$\begin{aligned} \sum_{j=0}^{n-1} (2j + 1) &= \left[\sum_{j=0}^{n-2} (2j + 1) \right] + [2(n - 1) + 1] \\ &= (n - 1)^2 + 2(n - 1) + 1 = ((n - 1) + 1)^2 = n^2. \quad \square \end{aligned}$$

0.0.5 Remark. Although the formula in Problem 0.0.4 is verified using induction, this approach feels unsatisfying. It is overly formal and does not illuminate the underlying reason the equation holds. Figure 0.2 offers a more intuitive way to understand the sum.

0.0.6 Problem. For every $n \in \mathbb{N}$ satisfying $n > 3$, prove that $2^n \geq n^2$.

Inductive solution. For all $n \in \mathbb{N}$ such that $n > 1$, we first prove, by induction, that $n^2 \geq 2$. When $n = 2$, we have $2^2 = 4 > 2$, so the base case holds. Assuming that $n^2 - 2 \geq 0$, the induction step is

$$(n + 1)^2 - 2 = (n^2 + 2n + 1) - 2 \geq 2n + 1 \geq 5 \geq 0.$$

For all $n \in \mathbb{N}$ such that $n \geq 4$, we now prove, via induction on n , that $2^n \geq n^2$. When $n = 4$, we have $2^4 = 16 = 4^2$, so the base case holds. Assuming $2^n - n^2 \geq 0$, the induction step is

$$\begin{aligned} 2^{n+1} - (n + 1)^2 &= 2(2^n) - n^2 - 2n - 1 \\ &\geq 2(n^2) - n^2 - 2n - 1 = (n - 1)^2 - 2. \end{aligned}$$

As the first paragraph establishes that $(n - 1)^2 - 2 \geq 0$, we deduce that $2^{n+1} - (n + 1)^2 \geq 0$ as required. $\square$

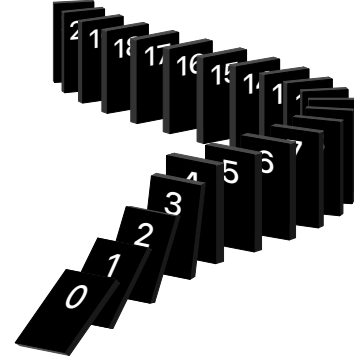


Figure 0.1: Induction is like toppling dominoes. Pushing the zeroth domino corresponds to the base case. Each subsequent domino being knocked over by its predecessor represents the induction step.

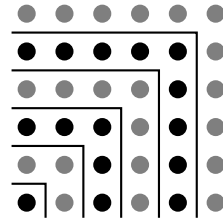


Figure 0.2: Sum of odd integers

0.1 Peano Arithmetic

How can the nonnegative integers be equipped with addition and multiplication? We begin with a preliminary technical observation.

0.1.0 Proposition. *For all maps $\varphi: \mathbb{N} \rightarrow \mathbb{N}$ and $\psi: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$, there exists a unique function $\theta: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ such that $\theta(m, 0) = \varphi(m)$ and $\theta(m, S(n)) = \psi(\theta(m, n), m)$ for all $m, n \in \mathbb{N}$.*

This proposition is used solely in definitions of addition and multiplication.

Proof. Fix $m \in \mathbb{N}$. The principle of induction determines a function $\beta_m: \mathbb{N} \rightarrow \mathbb{N}$ such that $\beta_m(0) := \varphi(m)$ and $\beta_m(S(n)) := \psi(\beta_m(n), m)$ for all $n \in \mathbb{N}$. Define the function $\theta: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ by $\theta(m, n) := \beta_m(n)$. By construction, θ satisfies the two conditions. Since the conditions determine all the outputs, the function θ is uniquely defined. $\square$

Addition of nonnegative integers arises as a special case.

0.1.1 Definition. The unique function $\alpha: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ such that $\alpha(m, 0) = m$ and $\alpha(m, S(n)) = S(\alpha(m, n))$ for all $m, n \in \mathbb{N}$ is called *addition* and invariably denoted by $m + n := \alpha(m, n)$.

Apply Proposition 0.1.0 when $\varphi = \text{id}_{\mathbb{N}}$ and $\psi(m, n) = S(m)$. As $1 := S(0)$, the definition of addition gives $m + 1 = m + S(0) = S(m + 0) = S(m)$ for all $m \in \mathbb{N}$.

0.1.2 Lemma. *For all $m \in \mathbb{N}$, we have $0 + m = m$.*

The distinguished element $0 \in \mathbb{N}$ is the additive identity.

Proof. Consider the subset $\mathcal{X} := \{m \in \mathbb{N} \mid 0 + m = m\}$. By definition, addition gives $0 + 0 = \alpha(0, 0) = 0$, so $0 \in \mathcal{X}$. When $n \in \mathcal{X}$, the definition of addition implies that

$$0 + S(n) = \alpha(0, S(n)) = S(\alpha(0, n)) = S(0 + n) = S(n)$$

so $S(n) \in \mathcal{X}$. Thus, the principle of induction yields $\mathcal{X} = \mathbb{N}$. $\square$

0.1.3 Lemma. *For all $m, n \in \mathbb{N}$, we have $S(m) + n = S(m + n)$.*

Proof. Fix $m \in \mathbb{N}$ and consider $\mathcal{X} := \{n \in \mathbb{N} \mid S(m) + n = S(m + n)\}$. The definition of addition gives $S(m) + 0 = S(m) = S(m + 0)$, so $0 \in \mathcal{X}$. When $n \in \mathcal{X}$, the definition of addition implies that

$$S(m) + S(n) = S(S(m) + n) = S(S(m + n)) = S(m + S(n)),$$

so $S(n) \in \mathcal{X}$. Thus, the principle of induction yields $\mathcal{X} = \mathbb{N}$. $\square$

Multiplication of nonnegative integers is very similar.

0.1.4 Definition. The unique function $\mu: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ such that $\mu(m, 0) = 0$ and $\mu(m, S(n)) = \mu(m, n) + m$ for all $m, n \in \mathbb{N}$ is called *multiplication* and typically denoted by $m n := \mu(m, n)$.

Apply Proposition 0.1.0 when $\varphi(m) = 0$ and $\psi(m, n) = m + n$. The definition of multiplication gives $m 1 = m 0 + m = m$ for all $m \in \mathbb{N}$. Hence, the element $1 := S(0)$ is the multiplicative identity.

0.1.5 Lemma. *For all $m \in \mathbb{N}$, we have $0 m = 0$.*

Proof. Consider the subset $\mathcal{X} := \{m \in \mathbb{N} \mid 0 m = 0\}$. The definition of multiplication implies that $(0)(0) = 0$, so $0 \in \mathcal{X}$. When $n \in \mathcal{X}$, the definition of multiplication and the definition of addition show that $0 S(n) = 0 n + 0 = 0 + 0 = 0$, so $S(n) \in \mathcal{X}$. The principle of induction again yields $\mathcal{X} = \mathbb{N}$. $\square$

We now prove the key properties of addition and multiplication.

0.1.6 Theorem. For all $k, m, n \in \mathbb{N}$, we have

$$\begin{array}{ll} \text{(associativity of addition)} & (k + m) + n = k + (m + n), \\ \text{(commutativity of addition)} & m + n = n + m, \\ \text{(distributivity)} & k(m + n) = km + kn, \\ \text{(associativity of multiplication)} & k(mn) = (km)n, \\ \text{(commutativity of multiplication)} & mn = nm. \end{array}$$

Proof. Fix $k, m \in \mathbb{N}$ and consider the subset

$$\mathcal{X}_0 := \{n \in \mathbb{N} \mid k + (m + n) = (k + m) + n\}.$$

The definition of addition gives $k + (m + 0) = k + m = (k + m) + 0$, so $0 \in \mathcal{X}_0$. When $n \in \mathcal{X}_0$, the definition of addition implies that

$$\begin{aligned} k + (m + S(n)) &= k + S(m + n) = S(k + (m + n)) \\ &= S((k + m) + n) = (k + m) + S(n), \end{aligned}$$

so $S(n) \in \mathcal{X}_0$. Thus, the principle of induction yields $\mathcal{X}_0 = \mathbb{N}$.

Fix $m \in \mathbb{N}$ and consider $\mathcal{X}_1 := \{n \in \mathbb{N} \mid m + n = n + m\}$. The definition of addition and Lemma 0.1.2 give $m + 0 = m = 0 + m$, so $0 \in \mathcal{X}_1$. When $n \in \mathcal{X}_1$, Lemma 0.1.3 and the definition of addition imply that $S(n) + m = S(n + m) = S(m + n) = m + S(n)$, so $S(n) \in \mathcal{X}_1$. Thus, the principle of induction yields $\mathcal{X}_1 = \mathbb{N}$.

Fix $k, m \in \mathbb{N}$ and consider $\mathcal{X}_2 := \{n \in \mathbb{N} \mid k(m + n) = km + kn\}$. The definitions of addition and multiplication give

$$k(m + 0) = km = km + 0 = km + k0,$$

so $0 \in \mathcal{X}_2$. When $n \in \mathcal{X}_2$, the definitions of addition and multiplication, together with the associativity of addition, imply that

$$\begin{aligned} k(m + S(n)) &= k S(m + n) = k(m + n) + k \\ &= (km + kn) + k = km + (kn + k) = km + k S(n), \end{aligned}$$

so $S(n) \in \mathcal{X}_2$. Thus, the principle of induction yields $\mathcal{X}_2 = \mathbb{N}$.

Fix $k, m \in \mathbb{N}$ and consider $\mathcal{X}_3 := \{n \in \mathbb{N} \mid k(mn) = (km)n\}$. The definition of multiplication gives $k(m0) = k0 = 0 = (km)0$, so $0 \in \mathcal{X}_3$. When $n \in \mathcal{X}_3$, the definition of multiplication, together with distributivity, imply that

$$\begin{aligned} k(m S(n)) &= k(mn + m) = k(mn) + km \\ &= (km)n + (km) = (km)(n + 1) = (km)S(n), \end{aligned}$$

so $S(n) \in \mathcal{X}_3$. Thus, the principle of induction yields $\mathcal{X}_3 = \mathbb{N}$.

We claim that $S(m)n = mn + n$ for all $m, n \in \mathbb{N}$. Fix $m \in \mathbb{N}$ and consider $\mathcal{X}_4 := \{n \in \mathbb{N} \mid S(m)n = mn + n\}$. The definitions of multiplication and addition give $S(m)0 = 0 = m0 + 0$, so $0 \in \mathcal{X}_4$. When $n \in \mathcal{X}_4$, the definitions of multiplication and addition, the properties of addition, and distributivity imply that

$$\begin{aligned} S(m)S(n) &= S(m)n + S(m) = (mn + n) + (m + 1) \\ &= mn + m + n + 1 = m(n + 1) + (n + 1) = mS(n) + S(n), \end{aligned}$$

so $S(n) \in \mathcal{X}_4$. Thus, the principle of induction yields $\mathcal{X}_4 = \mathbb{N}$.

Lastly, fix $m \in \mathbb{N}$ and consider $\mathcal{X}_5 := \{n \in \mathbb{N} \mid mn = nm\}$. The definition of multiplication and Lemma 0.1.5 give $m0 = 0 = 0m$, so $0 \in \mathcal{X}_5$. When $n \in \mathcal{X}_5$, the claim and distributivity imply that

$$S(n)m = nm + m = mn + m = m(n + 1) = mS(n)$$

so $S(n) \in \mathcal{X}_5$. Thus, the principle of induction yields $\mathcal{X}_5 = \mathbb{N}$. $\square$

Hilbert's second problem asks for a proof that this arithmetic is consistent—free of any internal contradictions; see D. Hilbert, *Mathematical problems*, Bull. Amer. Math. Soc. 8 (1902), 437–479.

Compare this claim with Lemma 0.1.3.

0.2 Well-Ordering

How can the nonnegative integers be endowed with a total order? We record another feature of nonnegative integers before defining the canonical binary relation on them.

0.2.0 Lemma. *Let $0 \neq n \in \mathbb{N}$. For all $m \in \mathbb{N}$, we have $m \neq m + n$.*

Proof. Consider $\mathcal{X} := \{m \in \mathbb{N} \mid m \neq m + n \text{ for all } 0 \neq n \in \mathbb{N}\}$. Since $0 \neq n = 0 + n$, it follows that $0 \in \mathcal{X}$. When $m \in \mathcal{X}$, we claim that $S(m) \neq S(m) + n$. Commutativity of addition and the definition of addition give $S(m) + n = n + S(m) = S(n + m) = S(m + n)$. Hence, it suffices to show that $S(m) \neq S(m + n)$. Because the map $S: \mathbb{N} \rightarrow \mathbb{N}$ is injective, this relation is equivalent to $m \neq m + n$, so $S(m) \in \mathcal{X}$. Thus, the principle of induction yields $\mathcal{X} = \mathbb{N}$. $\square$

0.2.1 Definition. For all $m, n \in \mathbb{N}$, we say ‘ m is less than n ’, denoted by $m < n$, if there exists a nonzero $k \in \mathbb{N}$ such that $n = m + k$.

0.2.2 Remark. Since Lemma 0.1.2 shows that $n = 0 + n$ for all $n \in \mathbb{N}$, it follows that $0 \leq n$ and $n \geq 0$.

An important trichotomy arises from this definition.

0.2.3 Proposition. *For all $m, n \in \mathbb{N}$, exactly one of the following holds:*

$$m < n, \quad m = n, \quad \text{or} \quad m > n.$$

Proof. We first show that the relations are mutually exclusive.

(< and =) Suppose that $m < n$ and $m = n$. By definition, there exists a nonzero $k \in \mathbb{N}$ such that $m = n = m + k$, which contradicts Lemma 0.2.0.

(< and >) Suppose that $m < n$ and $m > n$. By definition, there exist nonzero $j, k \in \mathbb{N}$ such that $n = m + j$ and $m = n + k$. We deduce that $n = n + (j + k)$, which again contradicts Lemma 0.2.0.

(= and >) Suppose that $m = n$ and $m > n$. By symmetry, this is equivalent to the first case.

It remains to show that one of the relations always holds. Fix $m \in \mathbb{N}$ and consider $\mathcal{X} := \{n \in \mathbb{N} \mid m < n \text{ or } m = n \text{ or } m > n\}$. When $m = 0$, we have $0 \in \mathcal{X}$. When $m \neq 0$, we have $m = 0 + m$, so $0 < m$ and $0 \in \mathcal{X}$. When $n \in \mathcal{X}$, there are three cases:

(<) Suppose that $m < n$. By definition, there exists a nonzero $k \in \mathbb{N}$ such that $n = m + k$. The definition of addition gives

$$S(n) = S(m + k) = m + S(k),$$

so $m < S(n)$ and $S(n) \in \mathcal{X}$.

(=) Suppose that $m = n$. Since $S(n) = n + 1 = m + 1$, it follows that $m < S(n)$, so $S(n) \in \mathcal{X}$.

(>) Suppose that $m > n$. By definition, there exists a nonzero $k \in \mathbb{N}$ such that $m = n + k$. When $k = 1$, we have $m = n + 1 = S(n)$,

In mathematics, the adjective “canonical” indicates an object that is prescribed by or arises naturally from a definition or construction.

When $m < n$, we also say ‘ n is greater than m ’ and write $n > m$. We use the notation $m \leq n$ or $n \geq m$ whenever $m = n$ or $m < n$.

We have $6 < 10$ because $10 = 6 + 4$.

The binary relation $<$ defines a *strict total order* on $\mathbb{N}$; for all $k, m, n \in \mathbb{N}$, it has the following properties.

(*Irreflexive*) No element is less than itself: $n \not< n$.

(*Asymmetric*) The inequality $m < n$ implies that $n \not< m$.

(*Transitive*) The inequalities $k < m$ and $m < n$ give $k < n$.

(*Complete*) When $m \neq n$, either $m < n$ or $m > n$.

so $S(n) \in \mathcal{X}$. When $k \neq 1$, there is a nonzero $j \in \mathbb{N}$ such that $k = S(j)$. It follows that

$$m = n + k = n + S(j) = n + (j + 1) = (n + 1) + j = S(n) + j,$$

so $S(n) < m$ and $S(n) \in \mathcal{X}$.

Thus, the principle of induction implies that $\mathcal{X} = \mathbb{N}$. $\square$

0.2.4 Corollary. *For all $k, m, n \in \mathbb{N}$ such that $k \neq 0$, the inequality $m < n$ holds if and only if the inequality $km < kn$ holds.*

Proof. Suppose that $m < n$. By definition, there exists a nonzero $j \in \mathbb{N}$ such that $n = m + j$. Distributivity gives

$$kn = k(m + j) = km + kj.$$

Since kj is nonzero, we deduce that $km < kn$.

Conversely, suppose that $km < kn$. Consider two cases.

(=) Suppose that $m = n$. It follows that $km = kn$ contradicting the inequality $km < kn$.

(>) Suppose that $m > n$. The first paragraph shows that $km > kn$ which also contradicts the inequality $km < kn$.

From the trichotomy, we deduce that $m < n$. $\square$

0.2.5 Lemma. *For all $m, n \in \mathbb{N}$ such that $m < n$, we have $S(m) \leq n$.*

Proof. Since $m < n$, there exists a nonzero $k \in \mathbb{N}$ such that $n = m + k$. When $k = 1$, we have $n = m + 1 = S(m)$. When $k > 1$, there exists a nonzero $j \in \mathbb{N}$ such that $k = 1 + j$. It follows that $n = m + 1 + j$ and $S(m) = m + 1 < n$. $\square$

Using the total order on nonnegative integers, the principle of induction has another fundamental reformulation.

0.2.6 Theorem (Well-ordering of $\mathbb{N}$). *Every nonempty subset of the set of nonnegative integers contains a unique least element.*

Proof. Let $\mathcal{X}$ be a nonempty subset of nonnegative integers.

(Uniqueness) Suppose that x and y are both least elements of $\mathcal{X}$.

Since $x \leq y$ and $y \leq x$, the trichotomy implies that $x = y$.

(Existence) Suppose that subset $\mathcal{X}$ has no least element. Consider the subset $\mathcal{Y} := \{n \in \mathbb{N} \mid n \leq x \text{ for all } x \in \mathcal{X}\}$. Remark 0.2.2 proves that $0 \in \mathcal{Y}$. Assume that $n \in \mathcal{Y}$. As $\mathcal{X}$ has no least element, it follows that $n \notin \mathcal{X}$ and $n < x$ for all $x \in \mathcal{X}$. Lemma 0.2.5 shows that $S(n) = n + 1 \leq x$ for all $x \in \mathcal{X}$, so $S(n) \in \mathcal{Y}$. The principle of induction establishes that $\mathcal{Y} = \mathbb{N}$. However, this would imply that $\emptyset = \mathcal{X} \cap \mathcal{Y} = \mathcal{X}$ which contradicts the hypothesis that $\mathcal{X}$ is nonempty. Therefore, we conclude that every nonempty subset of nonnegative integers has a least element. $\square$

In other words, the binary relation $<$ on $\mathbb{N}$ is compatible with multiplication. Problem 0.3.18 shows that it is also compatible with addition.

The set $\mathbb{N}$ of nonnegative integers represents the smallest infinite ordinal.

0.3 Exercises

0.3.1 Problem. Determine which of the following statements are true. If a statement is false, provide a counterexample and, if possible, salvage it by modifying the statement so that it becomes true.

- i. The successor function $S: \mathbb{N} \rightarrow \mathbb{N}$ is surjective.
- ii. The induction step may hold even if the base case is false.
- iii. In the induction step, assuming that $P(n)$ holds means that one has already proven that $P(n) = \text{true}$.
- iv. For all $n \in \mathbb{N}$, there exists an $m \in \mathbb{N}$ such that $m + n = 0$.
- v. For all $n \in \mathbb{N}$, we have $1n = n$.
- vi. The number 1 is the smallest element in the set $\mathbb{N}$.
- vii. The set $\mathbb{N}$ has a unique largest element.

0.3.2 Problem. For all $n \in \mathbb{N}$, prove that the equation

$$\sum_{k=0}^n \frac{1}{(k+1)(k+2)} = \frac{n+1}{n+2}$$

holds in two ways.

- i. Verify this equation by induction on n .
- ii. Derive this equation using partial fractions.

0.3.3 Problem. For all $n \in \mathbb{N}$, verify each of the following identities

$$\begin{aligned} \sum_{k=0}^n k^2 &= \frac{1}{6}n(n+1)(2n+1), & \sum_{k=0}^n k^4 &= \frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1), \\ \sum_{k=0}^n k^3 &= \frac{1}{4}n^2(n+1)^2, & \sum_{k=0}^n k^5 &= \frac{1}{12}n^2(n+1)^2(2n^2+2n-1). \end{aligned}$$

0.3.4 Problem. For all $n \in \mathbb{N}$, verify each of the following identities

$$\begin{aligned} \sum_{k=0}^n 2^k &= 2^{n+1} - 2, & \sum_{k=0}^n k^2 2^k &= 2^{n+1}(n^2 - 3n + 3) - 6, \\ \sum_{k=0}^n k 2^k &= 2^{n+1}(n - 1) - 2, & \sum_{k=0}^n k^3 2^k &= 2^{n+1}(n^3 - 3n^2 + 9n - 13) + 26. \end{aligned}$$

0.3.5 Problem. For all $n \in \mathbb{N}$, verify that

$$\sum_{j=0}^n j^3 = \left(\sum_{j=0}^n j \right)^2$$

0.3.6 Problem. For all $m \in \mathbb{Z}$ and all $n \in \mathbb{N}$, confirm that

$$\sum_{k=0}^n (k^2 + mk + 1)k! = (m+n)(n+1)! + (1-m)$$

0.3.7 Problem. For all $n \in \mathbb{N}$, prove that either $n = 0$ or there exists $k \in \mathbb{N}$ such that $n = S(k)$.

0.3.8 Problem. Establish the following variant on the principle of induction.

To verify that a property $P(n)$ holds for all $n \in \mathbb{N}$, it is enough to prove that

(Base case) $P(0)$ holds, and

(Induction step) for all $n \in \mathbb{N}$, the assumption that the property $P(k)$ holds for all $k \leq n$ implies that the property $P(n+1)$ holds.

0.3.9 Problem. The sequence of *square triangular numbers* is defined, for all $k \geq 2$, by $N_0 := 0$, $N_1 := 1$, and $N_k := 34N_{k-1} - N_{k-2} + 2$. The first few terms are 0, 1, 36, 1225, 41616, 1413721, 48024900, ...

- i. Prove that $N_{k-1}N_{k+1} = (N_k - 1)^2$ for all $k \geq 1$.

ii. Verify that

$$N_k = \left[\frac{(3 + 2\sqrt{2})^k - (3 - 2\sqrt{2})^k}{4\sqrt{2}} \right]^2.$$

0.3.10 Problem. The *Fibonacci numbers* are defined by $F_0 := 0$, $F_1 := 1$, and $F_{n+2} := F_{n+1} + F_n$ for all $n \in \mathbb{N}$. The first few terms are 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, ...

i. For all $n \in \mathbb{N}$, verify that $F_{n+1}^2 = F_{n+2} F_n + (-1)^n$.

ii. For all $n \in \mathbb{N}$, confirm that $\sum_{k=0}^n F_k^2 = F_n F_{n+1}$.

0.3.11 Problem. For all $k, m, n \in \mathbb{N}$, show that $k + m = k + n$ if and only if $m = n$.

0.3.12 Problem. For all $m, n \in \mathbb{N}$, establish that $m + n = 0$ if and only if $m = n = 0$.

0.3.13 Problem. For all $k, m, n \in \mathbb{N}$ such that $k \neq 0$, demonstrate that $km = kn$ if and only if $m = n$.

0.3.14 Problem. For all $m, n \in \mathbb{N}$, show that $mn = 0$ if and only if $m = 0$ or $n = 0$.

0.3.15 Problem. For all $m, n \in \mathbb{N}$, prove that $mn = 1$ if and only if $m = n = 1$.

0.3.16 Problem. For nonnegative integers, formulate the definition for exponentiation and prove that, for all $k, m, n \in \mathbb{N}$, we have

$$m^0 = 1, \quad (mn)^k = m^k n^k, \quad m^{k+n} = m^k m^n, \quad \text{and} \quad (m^n)^k = m^{nk}.$$

0.3.17 Problem. For all $k, m, n \in \mathbb{N}$, establish the following:

i. When $k \leq m$ and $m \leq n$, we have $k \leq n$.

ii. When $k < m$ and $m \leq n$, we have $k < n$.

iii. When $k \leq m$ and $m < n$, we have $k < n$.

0.3.18 Problem. For all $k, m, n \in \mathbb{N}$, prove that $m < n$ if and only if $k + m < k + n$.

0.3.19 Problem. Establish that any nonempty subset of nonnegative integers that is bounded above has a unique greatest element (with respect to $\leq$).

0.3.20 Problem. For all $m, n \in \mathbb{N}$ such that $n \neq 0$, prove that there exists $k \in \mathbb{N}$ such that $m < kn$.