

Problems 06

Due: Friday, 23 October 2026 before 23:59 ET

P6.1. Let $\mathbb{F}_4$ denote the subset of all (2×2) -matrices having the form

$$\begin{bmatrix} a & b \\ b & a + b \end{bmatrix},$$

where a and b are ring elements in the quotient $\mathbb{Z}/\langle 2 \rangle$.

- i. Show that $\mathbb{F}_4$ is a subring of the ring $\text{Mat}(2, \mathbb{Z}/\langle 2 \rangle)$ consisting of all (2×2) -matrices with entries in the quotient $\mathbb{Z}/\langle 2 \rangle$.
- ii. Verify that $\mathbb{F}_4$ is a commutative ring.
- iii. Demonstrate that any nonzero element in $\mathbb{F}_4$ has a multiplicative inverse.

P6.2. i. Let $R := \mathbb{Z}/\langle 6 \rangle$. For the polynomials

$$f = x^5 + 5x^4 + 4x^3 + 3x^2 + 1 \quad \text{and} \quad g = 3x^2 + x + 2$$

in $R[x]$, find a quotient and remainder for division of $3^4 f$ by g .

- ii. Let K be a field. Consider polynomials $f, g \in K[x]$ such that $\deg(g) > 0$. Confirm that there exist unique polynomials $h_0, h_1, \dots, h_d \in K[x]$ such that

$$f = h_0 + h_1 g + h_2 g^2 + h_3 g^3 + \dots + h_d g^d$$

where $\deg(h_j) < \deg(g)$ or $h_j = 0$ for all $0 \leq j \leq d$.

P6.3. Let R be a commutative ring. The *derivative operator* $D: R[x] \rightarrow R[x]$ is defined, for every polynomial $f := a_m x^m + a_{m-1} x^{m-1} + \dots + a_1 x + a_0 \in R[x]$, by

$$D(f) := (m a_m) x^{m-1} + ((m-1) a_{m-1}) x^{m-2} + \dots + a_1.$$

- i. Prove that the operator D is R -linear: for all $r, s \in R$ and all $f, g \in R[x]$, we have

$$D(rf + sg) = rD(f) + sD(g).$$

- ii. Prove that the operator D satisfies the product rule: for all $f, g \in R[x]$, we have

$$D(fg) = D(f)g + fD(g).$$

- iii. Consider $f \in R[x]$ and let $b \in R$ be root of f having multiplicity k with $k \geq 1$. Prove that b is also a root of the derivative $D(f)$ having multiplicity at least $k - 1$. Moreover, when the product $k 1_R$ is invertible in R , show that b is a root of the derivative $D(f)$ having multiplicity $k - 1$.