

Solutions 02

P2.1. The *Fibonacci numbers* are defined by $F_0 := 0$, $F_1 := 1$, and

$$F_{n+2} := F_{n+1} + F_n \quad \text{for all } n \in \mathbb{N}.$$

The first few terms are 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, ...
For all $n \in \mathbb{N}$, verify that $F_{n+1}^2 - F_{n+2} F_n = (-1)^n$.

Solution. We proceed by induction on n . When $n = 0$ and $n = 1$, we have

$$F_1^2 - F_2 F_0 - (-1)^0 = (1)^2 - (1)(0) - 1 = 0 \quad \text{and}$$

$$F_2^2 - F_3 F_1 - (-1)^1 = (1)^2 - (2)(1) + 1 = 0,$$

so the base cases hold. Assume that

$$F_{n+1}^2 - F_{n+2} F_n - (-1)^n = 0.$$

Using the defining recurrence relation twice and the induction hypothesis gives

$$\begin{aligned} F_{n+2}^2 - F_{n+3} F_{n+1} - (-1)^{n+1} &= F_{n+2}^2 - (F_{n+2} + F_{n+1})F_{n+1} + (-1)^n \\ &= F_{n+2}(F_{n+2} - F_{n+1}) - F_{n+1}^2 + (-1)^n \\ &= -(-F_{n+2} F_n + F_{n+1}^2 - (-1)^n) = -(0) = 0, \end{aligned}$$

which completes the induction step. □

P2.2. Establish the cancellation law for addition of nonnegative integers: for all $k, m, n \in \mathbb{N}$, show that the equation $m + k = n + k$ implies that $m = n$.

Solution. Fix $m, n \in \mathbb{N}$ and consider the subset

$$\mathcal{X} := \{k \in \mathbb{N} \mid m + k = n + k \text{ implies that } m = n\}.$$

The definition of addition gives $m + 0 = m$ and $n + 0 = n$. Hence, the equation $m + 0 = n + 0$ implies that $m = m + 0 = n + 0 = n$, so $0 \in \mathcal{X}$. Suppose that $k \in \mathcal{X}$. If $m + S(k) = n + S(k)$, then the definition of addition gives

$$S(m + k) = m + S(k) = n + S(k) = S(n + k).$$

The Peano Axiom (C1) asserts that the successor function S is injective, so we deduce that $m + k = n + k$. Since $k \in \mathcal{X}$, it follows that $m = n$ and $S(k) \in \mathcal{X}$. Thus, the principle of induction yields $\mathcal{X} = \mathbb{N}$. □

P2.3. Use the well-ordering of the set $\mathbb{N}$ to demonstrate that every nonempty subset of the nonnegative integers that is bounded above has a unique greatest element.

Solution. Let $\mathcal{X}$ be a nonempty subset of nonnegative integers that is bounded above. Consider the associated subset $\mathcal{Y} \subset \mathbb{N}$ consisting of all upper bounds for $\mathcal{X}$;

$$\mathcal{Y} := \{n \in \mathbb{N} \mid x \leq n \text{ for all } x \in \mathcal{X}\}.$$

The set $\mathcal{Y}$ is nonempty because $\mathcal{X}$ is bounded above. As $\mathcal{X} \neq \emptyset$, we see that $0 \notin \mathcal{Y}$. The well-ordering of $\mathbb{N}$ implies that the subset $\mathcal{Y}$ has a unique least element m .

Suppose that m is not in $\mathcal{X}$. Since $m \neq 0$, there exists a nonnegative integer n such that $S(n) = m$. For every $x \in \mathcal{X}$, we have $x \leq m$ because $m \in \mathcal{Y}$, while $x \neq m$ by supposition. Thus, we would have $x < m$. Hence, there would exist a nonzero $k \in \mathbb{N}$ such that $m = x + k$. If $k = 1$, then we would have $n + 1 = S(n) = m = x + 1$, so $n = x$. If $k > 1$, then there would exist a nonzero $j \in \mathbb{N}$ such that $k = 1 + j$. It would follow that $n + 1 = S(n) = m = x + k = x + j + 1$, so $n = x + j$ and $x < n$. In either case, we would have $x \leq n$, so n would be an upper bound for $\mathcal{X}$ and $n \in \mathcal{Y}$. However, the inequality $n < m$ would contradict the fact that m is the least element in $\mathcal{Y}$. Thus, we deduce that m is in $\mathcal{X}$.

Since the least upper bound m for the set $\mathcal{X}$ also belongs to $\mathcal{X}$, it is the greatest element in $\mathcal{X}$. □