

Solutions 05

P5.1. Solve the system of linear equations $\mathbf{A}\mathbf{X} = \mathbf{B}$ where

$$\mathbf{A} := \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{B} := \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

and the entries are

- i. in $\mathbb{Q}$,
- ii. in $\mathbb{Z}/\langle 2 \rangle$,
- iii. in $\mathbb{Z}/\langle 3 \rangle$, or
- iv. in $\mathbb{Z}/\langle 5 \rangle$.

Solution.

- i. Finding the reduced row echelon form of the augmented matrix

$$\begin{aligned} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 \\ -1 & 1 & 0 & 1 & 1 \end{array} \right] &\sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & -1 & -2 & 0 & 0 \\ 0 & 1 & 1 & 2 & 2 \end{array} \right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & -2 & -1 & 1 \\ 0 & 0 & 1 & 3 & 1 \end{array} \right] \\ &\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 5 & 3 \\ 0 & 0 & 1 & 3 & 1 \end{array} \right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 1 & 3/5 \end{array} \right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 6/5 \\ 0 & 1 & 0 & 0 & 8/5 \\ 0 & 0 & 1 & 0 & -4/5 \\ 0 & 0 & 0 & 1 & 3/5 \end{array} \right], \end{aligned}$$

we see that the unique rational solution is $\mathbf{X} = \frac{1}{5}[6 \ 8 \ -4 \ 3]^T$.

- ii. Finding the reduced row echelon form of the augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \end{array} \right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{array} \right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{array} \right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array} \right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array} \right],$$

we see that the unique solution with entries in $\mathbb{Z}/\langle 2 \rangle$ is $\mathbf{X} = [0 \ 0 \ 0 \ 1]^T$.

- iii. Finding the reduced row echelon form of the augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 2 & 1 \\ 1 & 2 & 2 & 1 & 1 \\ 2 & 1 & 0 & 1 & 1 \end{array} \right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 2 \end{array} \right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{array} \right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{array} \right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right],$$

we see that the unique solution with entries in $\mathbb{Z}/\langle 3 \rangle$ is $\mathbf{X} = [0 \ 1 \ 1 \ 0]^T$.

- iv. Finding the reduced row echelon form of the augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 4 & 1 \\ 1 & 4 & 4 & 1 & 1 \\ 4 & 1 & 0 & 1 & 1 \end{array} \right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 4 & 1 \\ 0 & 4 & 3 & 0 & 0 \\ 0 & 1 & 1 & 2 & 2 \end{array} \right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 4 & 1 \\ 0 & 0 & 3 & 4 & 1 \\ 0 & 0 & 1 & 3 & 1 \end{array} \right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 3 & 0 \\ 0 & 1 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 3 & 1 \end{array} \right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 3 & 0 \\ 0 & 1 & 0 & 4 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right],$$

we see that there is no solution with entries in $\mathbb{Z}/\langle 5 \rangle$. □

Remark. Since $5 \equiv 1 \pmod{2}$, it follows that $5^{-1} \begin{bmatrix} 6 & 8 & -4 & 3 \end{bmatrix}^T \equiv \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T \pmod{2}$. Since $5 \equiv 2 \pmod{3}$, we have $5^{-1} \equiv 2 \pmod{3}$ and $5^{-1} \begin{bmatrix} 6 & 8 & -4 & 3 \end{bmatrix}^T \equiv \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}^T \pmod{3}$.

P5.2. Demonstrate that the equation $x^6 + y^{12} = 705$ has no integer solutions.

Solution. If the equation had an integer solution, then it would also have a solution modulo 7. Reducing modulo 7 gives $x^6 + y^{12} \equiv 5 \pmod{7}$. Fermat's Little Theorem gives

$$x^6 \equiv \begin{cases} 0 \pmod{7} & \text{if } x \equiv 0 \pmod{7}, \\ 1 \pmod{7} & \text{if } x \not\equiv 0 \pmod{7}, \end{cases} \quad \text{and} \quad y^{12} = (y^6)^2 \equiv \begin{cases} 0 \pmod{7} & \text{if } y \equiv 0 \pmod{7}, \\ 1 \pmod{7} & \text{if } y \not\equiv 0 \pmod{7}. \end{cases}$$

It follows that

$$x^6 + y^{12} \equiv \begin{cases} 0 \pmod{7} & \text{if } x \equiv 0 \pmod{7} \text{ and } y \equiv 0 \pmod{7}, \\ 1 \pmod{7} & \text{if } x \not\equiv 0 \pmod{7} \text{ and } y \equiv 0 \pmod{7}, \\ 1 \pmod{7} & \text{if } x \equiv 0 \pmod{7} \text{ and } y \not\equiv 0 \pmod{7}, \\ 2 \pmod{7} & \text{if } x \not\equiv 0 \pmod{7} \text{ and } y \not\equiv 0 \pmod{7}. \end{cases}$$

We deduce that $x^6 + y^{12} \not\equiv 5 \pmod{7}$. Since the congruence has no solutions, the original equation has no integer solutions. $\square$

P5.3. Determine whether the set $\mathbb{R} \cup \{\infty\}$ with addition and multiplication defined, for all $x, y \in \mathbb{R} \cup \{\infty\}$, by $x \boxplus y := \min(x, y)$ and $x \boxtimes y := x + y$, forms a commutative ring. Otherwise identify all of the defining properties that do hold and all that fail to hold.

Solution. The triple $(\mathbb{R} \cup \{\infty\}, \boxplus, \boxtimes)$ is not a commutative ring. The associativity of addition, the commutativity of addition, the existence of an additive identity, distributivity, the associativity of multiplication, the commutativity of multiplication, and the existence of a multiplicative identity hold. Indeed, for all $x, y, z \in \mathbb{R} \cup \{\infty\}$, we have

$$(x \boxplus y) \boxplus z = \min(\min(x, y), z) = \min(x, y, z) = \min(x, \min(y, z)) = x \boxplus (y \boxplus z),$$

$$x \boxplus y = \min(x, y) = \min(y, x) = y \boxplus x,$$

$$x \boxplus \infty = \min(x, \infty) = x,$$

$$x \boxtimes (y \boxplus z) = x + \min(y, z) = \min(x + y, x + z) = (x \boxtimes y) \boxplus (x \boxtimes z),$$

$$(x \boxtimes y) \boxtimes z = (x + y) + z = x + (y + z) = x \boxtimes (y \boxtimes z),$$

$$x \boxtimes y = x + y = y + x = y \boxtimes x, \quad \text{and}$$

$$x \boxtimes 0 = x + 0 = x,$$

The element ∞ is the additive identity and the element 0 is the multiplicative identity. The existence of an additive inverse does not hold: for all $x, y \in \mathbb{R} \cup \{\infty\}$, we have

$$x \boxplus y = \infty \iff \min(x, y) = \infty \iff x = y = \infty.$$

In other words, only ∞ has an additive inverse. $\square$

Remark. This algebraic structure is called the *min-plus algebra* or the *tropical semiring*.