

Problems 3

Due: Friday, 14 February 2025 before 11:59 EST

Students registered in MATH 413 should submit solutions to P3.1 and any other two problems, whereas students in MATH 813 should submit solutions to all five.

- P3.1.** Let $I := \langle xy - wz, wy - z^2, w^2 - xz \rangle$ be an ideal in the polynomial ring $\mathbb{Q}[w, x, y, z]$.
- Find (without using computer software) the reduced Gröbner basis of I with respect to the graded reverse lexicographic order and $w > x > y > z$.
 - Find (without using computer software) the reduced Gröbner basis of I with respect to the lexicographic order and $w > x > y > z$.
 - (*Bonus*) The ideal I has 8 distinct leading term ideals. Can you exhibit these eight monomial ideals?
- P3.2.** Fix the lexicographic order on the ring $S := \mathbb{K}[x_1, x_2, \dots, x_n]$ where $x_1 > x_2 > \dots > x_n$. Let $\mathbf{A} := [a_{j,k}]$ be an $(m \times n)$ -matrix with entries in the field $\mathbb{K}$. For all $1 \leq j \leq m$, let $f_j := a_{j,1}x_1 + a_{j,2}x_2 + \dots + a_{j,n}x_n$ be the linear polynomial determined by the j th row of the matrix $\mathbf{A}$. Suppose that $\mathbf{B}$ is the reduced row echelon matrix associated to $\mathbf{A}$ and let $g_1, g_2, \dots, g_r$ be the linear polynomials determined by the nonzero rows in $\mathbf{B}$.
- Prove that $\langle f_1, f_2, \dots, f_m \rangle = \langle g_1, g_2, \dots, g_r \rangle$.
 - Show that $g_1, g_2, \dots, g_r$ form a Gröbner basis of the ideal $\langle f_1, f_2, \dots, f_m \rangle$.
 - Explain why $g_1, g_2, \dots, g_r$ is the reduced Gröbner basis for the ideal $\langle f_1, f_2, \dots, f_m \rangle$.
- P3.3.** Suppose that the numbers a, b, c , and d satisfy the equations
- $$\begin{aligned} a + b + c + d &= 3, & a^3 + b^3 + c^3 + d^3 &= 237, \\ a^2 + b^2 + c^2 + d^2 &= 87, & a^4 + b^4 + c^4 + d^4 &= 3123. \end{aligned}$$
- Prove that $a^5 + b^5 + c^5 + d^5 = 13893$.
 - Show that $a^6 + b^6 + c^6 + d^6 \neq 17$.
 - What are $a^6 + b^6 + c^6 + d^6$ and $a^7 + b^7 + c^7 + d^7$?
- P3.4.** The *Whitney umbrella* is the image of the map $\rho: \mathbb{A}^2 \rightarrow \mathbb{A}^3$ defined by $(u, v) \mapsto (uv, v, u^2)$.
- Find the equation(s) for the smallest affine subvariety in $\mathbb{A}^3$ containing the Whitney umbrella.
 - Show that the parametrization fills up this affine subvariety over $\mathbb{C}$ but not over $\mathbb{R}$. Over $\mathbb{R}$, exactly what points are omitted?
 - Demonstrate that the parameters u and v are not always uniquely determined by a point in $\mathbb{A}^3$. Find the points where uniqueness fails.
- P3.5.** Consider the ideal $I := \langle 9x^2 - xy - 180, 81x^2 - 20xy - y^2 - 1620 \rangle$ in $\mathbb{Q}[x, y]$.
- Find Gröbner basis for $I \cap \mathbb{Q}[x]$ and $I \cap \mathbb{Q}[y]$.
 - Find all solutions to $9x^2 - xy = 180$ and $81x^2 - 20xy - y^2 = 1620$ in $\mathbb{C}$.
 - Which of the solutions are rational?
 - What is the smallest field $\mathbb{K}$ containing $\mathbb{Q}$ such that all solutions lie in $\mathbb{K}$?