

Problems 4

Due: Friday, 7 March 2025 before 23:59 EST

Students registered in MATH 413 should submit solutions to any three problems, whereas students in MATH 813 should submit solutions to all five.

P4.1. Assume that $\mathbb{K}$ is an algebraically closed field. Identify affine space $\mathbb{A}^9(\mathbb{K})$ with the space of (3×3) -matrices $\mathbf{A} = [a_{j,k}]$. Let $\rho: \mathbb{A}^9(\mathbb{K}) \dashrightarrow \mathbb{A}^9(\mathbb{K})$ be the rational map defined by

$$\mathbf{A} \mapsto \mathbf{A} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{A}^{-1}.$$

- i. Find equations for the smallest affine subvariety X containing the image of ρ .
- ii. Show that X is the set of all nilpotent (3×3) -matrices.

P4.2. For any polynomial $f = a_\ell x^\ell + a_{\ell-1} x^{\ell-1} + \cdots + a_0 \in \mathbb{C}[x]$ where $a_\ell \neq 0$ and $\ell > 0$, the *discriminant* of f is defined to be

$$\text{disc}(f) = \frac{(-1)^{\ell(\ell-1)/2}}{a_\ell} \text{Res}(f, f'; x).$$

- i. The polynomial $f \in \mathbb{C}[x]$ is *separable* if it has only simple roots. Show that f is separable if and only if f is relatively prime to its derivative f' .
- ii. Prove that f has a multiple factor if and only if $\text{disc}(f) = 0$.
- iii. Does $6x^4 - 23x^3 + 32x^2 - 19x + 4$ have a multiple root in $\mathbb{C}$?
- iv. Compute the discriminant of the quadratic polynomial $f = ax^2 + bx + c$. Explain how your answer relates to the quadratic formula.

P4.3. Suppose that $f = a_m x^m + a_{m-1} x^{m-1} + \cdots + a_0$ and $g = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_0$. Consider the polynomial in two variables

$$\varphi(x, y) = \frac{f(x)g(y) - g(x)f(y)}{x - y} = \sum_{j=0}^{m-1} \sum_{k=0}^{m-1} c_{j,k} x^j y^k.$$

- i. When $m = 2$, show that $\text{Res}(f, g; x) = (-1) \det [c_{j,k}]$.
- ii. For any positive integer m , prove that $\text{Res}(f, g; x) = (-1)^{m(m-1)/2} \det [c_{j,k}]$.

P4.4. A subset U of the polynomial ring $S := \mathbb{K}[x_1, x_2, \dots, x_n]$ is *multiplicatively closed* if any product of elements of U is also in U (including the empty product 1).

- i. Let U be a multiplicatively closed subset of S . When an ideal I in S is maximal with respect to inclusion among all ideals not meeting U , show that I is prime.
- ii. Let J be any proper ideal in S . Show that the radical ideal $\sqrt{J}$ is the intersection of all prime ideals containing J .

P4.5. i. Find the minimal Gröbner basis for

$$\sqrt{\langle x^5 - 2x^4 + 2x^2 - x, x^5 - x^4 - 2x^3 + 2x^2 + x - 1 \rangle} \subset \mathbb{Q}[x].$$

- ii. Let $J = \langle xy, (x - y)x \rangle$. Describe $V(J)$ and show that $\sqrt{J} = \langle x \rangle$.