

Problems 5

Due: Friday, 21 March 2025 before 23:59 EST

Students registered in MATH 413 should submit solutions to any three problems, whereas students in MATH 813 should submit solutions to all five.

P5.1. Let $\mathbb{K}$ be a field.

- i. For any univariate polynomial $f := a_m x^m + a_{m-1} x^{m-1} + \cdots + a_1 x + a_0$ of degree m in the ring $\mathbb{K}[x]$, define its *homogenization* in the ring $\mathbb{K}[x, y]$ to be

$$f^h := a_m x^m + a_{m-1} x^{m-1} y + \cdots + a_1 x y^{m-1} + a_0 y^m.$$

Prove that the polynomial f has a root in the field $\mathbb{K}$ if and only if there exists a point (b, c) in $\mathbb{A}^2(\mathbb{K})$ such that $(b, c) \neq (0, 0)$ and $f^h(b, c) = 0$.

- ii. Assume that the field $\mathbb{K}$ is not algebraically closed. Exhibit a bivariate polynomial h in the ring $\mathbb{K}[x, y]$ such that the affine subvariety $V(h)$ in $\mathbb{A}^2(\mathbb{K})$ is just the origin $(0, 0)$.
- iii. Assume that the field $\mathbb{K}$ is not algebraically closed. For any positive integer n , demonstrate that there exists a polynomial f in the ring $\mathbb{K}[x_1, x_2, \dots, x_n]$ such that the affine subvariety $V(f)$ in $\mathbb{A}^n(\mathbb{K})$ is the origin $(0, 0, \dots, 0)$.
- iv. Assume that the field $\mathbb{K}$ is not algebraically closed. Prove that any affine subvariety $X = V(g_1, g_2, \dots, g_r)$ in $\mathbb{A}^n(\mathbb{K})$ can be defined by a single equation.

P5.2. For any ideal I in the ring $S := \mathbb{K}[x_1, x_2, \dots, x_n]$ and any polynomial f in S , the *saturation* of I with respect to f is the set

$$(I : f^\infty) := \{g \in S \mid \text{there exists a positive integer } m \text{ such that } f^m g \in I\}.$$

- i. Prove that $(I : f^\infty)$ is an ideal in the ring S .
- ii. Prove that there is an ascending chain of ideals $(I : f) \subseteq (I : f^2) \subseteq (I : f^3) \subseteq \cdots$.
- iii. For any positive integer ℓ , prove that we have the equality $(I : f^\infty) = (I : f^\ell)$ if and only if we have the equality $(I : f^\ell) = (I : f^{\ell+1})$.

P5.3. The ideals I and J in the ring $S := \mathbb{K}[x_1, x_2, \dots, x_n]$ are *comaximal* if $I + J = S$.

- i. Over an algebraically closed field, show that the ideals I and J are comaximal if and only if we have $V(I) \cap V(J) = \emptyset$. Without the algebraically closed hypothesis, show that this can be false.
- ii. When the ideals I and J are comaximal, show that $IJ = I \cap J$.
- iii. When the ideals I and J are comaximal, show that, for all positive integers i and j , the ideals I^i and J^j are comaximal.

- P5.4.** i. Consider the affine subvariety $X := V(xy - yz - y, x^2 - y^2 - z^2)$ in $\mathbb{A}^3$. Show that X is a union of three irreducible components. Describe them and find their prime ideals.
 ii. Show that the set of real points on the irreducible complex surface

$$V(x^2y - xz^2 + yz^2) \subset \mathbb{A}^3$$

is connected but is not equidimensional; it is the union of a closed curve and a closed surface in the induced Euclidean topology.

- P5.5.** i. Let I be a monomial ideal in the ring $S := \mathbb{K}[x_1, x_2, \dots, x_n]$. Suppose that the monomial x^u is a minimal generator of the ideal I and satisfies $x^u = x^{v_1} x^{v_2}$ for some relative prime monomials x^{v_1} and x^{v_2} . Show $I = (I + \langle x^{v_1} \rangle) \cap (I + \langle x^{v_2} \rangle)$.
 ii. Using part i, find an irredundant primary decomposition of the monomial ideal

$$\langle x^2y^2, x^2yz, xy^2z, x^2z^2, xyz^2, y^2z^2 \rangle .$$