

12 Power

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Category theory builds on itself: morphisms between categories are called functors and morphisms between functors are called natural transformations. The striking power of category theory emanates from this trinity of concepts.

12.0 Functoriality

Within category theory, objects should always be considered with a suitable notion of structure-preserving maps. Categories are themselves mathematical objects, so what are their morphisms?

12.0.1 Definition. A *functor* $F : \mathbf{C} \rightarrow \mathbf{D}$ consists of two assignments: for each object X in $\mathbf{C}$, an object $F(X)$ in $\mathbf{D}$, and for each morphism $f : X \rightarrow Y$ in $\mathbf{C}$, a morphism $F(f) : F(X) \rightarrow F(Y)$ in $\mathbf{D}$, such that
(*identity*) for each object X in $\mathbf{C}$, we have $F(\text{id}_X) = \text{id}_{F(X)}$, and
(*composition*) for any composable pair f and g of morphisms in $\mathbf{C}$, we have $F(g)F(f) = F(gf)$.

12.0.2 Example. There is a functor $P : \mathbf{Set} \rightarrow \mathbf{Set}$ that sends a set X to its power set $P(X)$ consisting of all subsets of X . This functor sends a map $f : X \rightarrow Y$ to the direct-image map $f_* : P(X) \rightarrow P(Y)$ defined, for all $X' \subseteq X$, by $f_*(X') := \{f(x) \mid x \in X'\} \subseteq Y$. $\diamond$

12.0.3 Example. A *forgetful functor* is a term used for any functor that forgets structure. The functor $U : \mathbf{Grp} \rightarrow \mathbf{Set}$ sends a group to its underlying set and a group homomorphism to its underlying map of sets. The functor $U : \mathbf{Top} \rightarrow \mathbf{Set}$ sends a space to its set of points. There are two natural forgetful functors $V : \mathbf{Graph} \rightarrow \mathbf{Set}$ and $E : \mathbf{Graph} \rightarrow \mathbf{Set}$ that send a graph to its vertex and edge sets, respectively. These mappings are functorial because in each case a morphism in the source category has an underlying map of sets. $\diamond$

12.0.4 Example. There are functors $\mathbf{Mod}_R \rightarrow \mathbf{Ab}$ and $\mathbf{CRng} \rightarrow \mathbf{Ab}$ that forget some, but not all, of the algebraic structure. The canonical functors $\mathbf{Ab} \rightarrow \mathbf{Grp}$ and $\mathbf{Field} \rightarrow \mathbf{CRng}$ may be regarded as forgetful. The latter two, but neither of the former, are injective on objects: a group is either abelian or not, but an abelian group might admit the structure of a ring in multiple ways. $\diamond$

12.0.5 Example. The chain rule articulates the functoriality of the derivative. Let $\mathbf{Euclid}_*$ be the category whose objects are pointed finite-dimensional Euclidean spaces $(\mathbb{R}^n, a)$ and whose morphisms are functions differentiable at a . The total derivative of a function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$, evaluated at $a \in \mathbb{R}^n$, gives rise to the Jacobian matrix. This defines a functor $D : \mathbf{Euclid}_* \rightarrow \mathbf{Mat}_{\mathbb{R}}$. On objects, D assigns a pointed Euclidean space to its dimensions. For a function

$g: \mathbb{R}^m \rightarrow \mathbb{R}^k$ that is differentiable at $f(a)$ and carries the designated basepoint $f(a) \in \mathbb{R}^m$ to $(gf)(a) \in \mathbb{R}^k$, the functoriality of D is equivalent to saying that the product of the Jacobian of f at a with the Jacobian of g at $f(a)$ equals the Jacobian of gf at a . This is the chain rule. $\diamond$

The functors appearing in Definition 12.0.1 are called *covariant* to distinguish them from another type of functor.

12.0.6 Definition. A *contravariant functor* F from the category $\mathbf{C}$ to the category $\mathbf{D}$ is a functor $F: \mathbf{C}^{\text{op}} \rightarrow \mathbf{D}$. For each $f: X \rightarrow Y$ in $\mathbf{C}$, there is a morphism $F(f): F(Y) \rightarrow F(X)$ in $\mathbf{D}$.

12.0.7 Example. The contravariant functor $P: \mathbf{Set}^{\text{op}} \rightarrow \mathbf{Set}$ sends a set X to its power set $P(X)$ and a map $f: X \rightarrow Y$ of sets to the inverse-image map $f^{-1}: P(Y) \rightarrow P(X)$ defined, for all $Y' \subseteq Y$, by $f^{-1}(Y') := \{x \in X \mid f(x) \in Y'\} \subseteq X$. $\diamond$

12.0.8 Example. There is a functor $(-)^*: \mathbf{Vect}_K^{\text{op}} \rightarrow \mathbf{Vect}_K$ that carries a K -vector space V to its dual space $V^* := \text{Hom}(V, K)$. A vector in V^* is a linear functional on V . This functor is contravariant because the linear map $\varphi: V \rightarrow W$ is sent to the linear map $\varphi^*: W^* \rightarrow V^*$ that pre-composes a linear functional $\omega: W \rightarrow K$ with φ to obtain the linear functional $(\omega \varphi): V \rightarrow K$. $\diamond$

12.0.9 Lemma. *Functors preserve isomorphisms*

Proof. Let $f: X \rightarrow Y$ be an isomorphism in a category $\mathbf{C}$ and let $g: Y \rightarrow X$ be its inverse. For any $F: \mathbf{C} \rightarrow \mathbf{D}$, functoriality implies that $F(g)F(f) = F(gf) = F(\text{id}_X) = \text{id}_{F(X)}$. By symmetry, we see that $F(g): F(Y) \rightarrow F(X)$ is the inverse of $F(f): F(X) \rightarrow F(Y)$. $\square$

12.0.10 Example. For any group G , let $\mathbf{BG}$ be its one-object category. A functor $X: \mathbf{BG} \rightarrow \mathbf{C}$ specifies an object X in $\mathbf{C}$ and a morphism $g_*: X \rightarrow X$ for any element g in G such that two conditions hold:

(*identity*) For the identity element e in G , we have $e_* = \text{id}_X$.

(*composition*) For any g and h in G , we have $h_*g_* = (hg)_*$.

Such functor defines an action of the group G on the object X in $\mathbf{C}$. When $\mathbf{C} = \mathbf{Set}$, the object X is endowed with a group action. When $\mathbf{C} = \mathbf{Vect}_K$, the object X is a representation of G . Because the elements g in G are isomorphisms in $\mathbf{BG}$, their images under any functor must also be isomorphisms in the target category. $\diamond$

12.0.11 Example. A category is *locally small* if between any pair of objects there is only a set's worth of morphisms. The category $\mathbf{CAT}$ has locally small categories as objects and functors between them as morphisms. The categories $\mathbf{Set}$, $\mathbf{Grp}$, $\mathbf{Poset}$ are objects in $\mathbf{CAT}$. The functor $(-)^{\text{op}}: \mathbf{CAT} \rightarrow \mathbf{CAT}$ defines a nontrivial automorphism of the category of categories. $\diamond$

In a locally small category $\mathbf{C}$, it is traditional to write $\text{Hom}_{\mathbf{C}}(X, Y)$ for the set of morphisms from X to Y .

12.1 Naturality

Categories and functors were first conceived as auxiliary concepts needed to give a precise meaning to the concept of naturality.

12.1.1 Definition. Let $F : \mathbf{C} \rightarrow \mathbf{D}$ and $G : \mathbf{C} \rightarrow \mathbf{D}$ be functors. A *natural transformation* $\alpha : F \Rightarrow G$ consists of a morphism $\alpha_X : F(X) \rightarrow G(X)$ in $\mathbf{D}$ for each object X in $\mathbf{C}$ such that, for any morphism $f : X \rightarrow Y$ in $\mathbf{C}$, the following square in $\mathbf{D}$

$$\begin{array}{ccc} F(X) & \xrightarrow{\alpha_X} & G(X) \\ F(f) \downarrow & & \downarrow G(f) \\ F(Y) & \xrightarrow{\alpha_Y} & G(Y) \end{array}$$

commutes.

12.1.2 Example. There is natural transformation $\eta : \text{id}_{\text{Set}} \Rightarrow P$ from the identity to the covariant power set functor where $\eta_X : X \rightarrow P(X)$ sends the element x in X to the singleton $\{x\}$ in $P(X)$. $\diamond$

12.1.3 Example. For any K -vector space V , the map $\text{ev} : V \rightarrow V^{**}$ that sends a vector v in V to the linear function $\text{ev}_v : V^* \rightarrow K$ defines a natural transformation from the identity functor on $\mathbf{Vect}_K$ to the double dual functor. To verify that the natural square commutes for any linear map $\varphi : V \rightarrow W$

$$\begin{array}{ccc} V & \xrightarrow{\text{ev}} & V^{**} \\ \varphi \downarrow & & \downarrow \varphi^{**} \\ W & \xrightarrow{\text{ev}} & W^{**}, \end{array}$$

it suffices to consider a generic vector v in V . The evaluation map $\text{ev}_{\varphi(v)} : W^* \rightarrow K$ carries a linear functional $f : W \rightarrow K$ to $f(\varphi(v))$. Example 12.0.8 shows that the morphism sends $\varphi^{**}(\text{ev}_v) : W^* \rightarrow K$ carries a linear function $f : W^* \rightarrow K$ to $f(\varphi(v))$. $\diamond$

A familiar isomorphism arising from the classification of finitely generated abelian groups is not natural.

12.1.4 Proposition. Let G be a finitely generated abelian group, and let $\tau(G)$ be its torsion subgroup. The isomorphism $G \cong \tau(G) \oplus (G/\tau(G))$ is not natural.

Proof. Suppose that the isomorphism $G \cong \tau(G) \oplus (G/\tau(G))$ were natural in G . The composite $G \rightarrow G/\tau(G) \rightarrow \tau(G) \oplus (G/\tau(G)) \cong G$ of the canonical quotient map, the canonical inclusion map, and this isomorphism would define a natural transformation on the identity functor on $\mathbf{Ab}^{\text{fg}}$. We claim that this is impossible.

We first show that any natural transformation $\alpha : \text{id}_{\mathbf{Ab}^{\text{fg}}} \rightarrow \text{id}_{\mathbf{Ab}^{\text{fg}}}$ is multiplication by some integer n . A homomorphism $\varphi : \mathbb{Z} \rightarrow G$ of

In practice, a natural transformation is defined by saying “the morphisms α_X are natural” which means that the collection defines a natural transformation. Although the correct choice of the source and target functors and the source and target categories may be implicit, the naturality condition refers to every object and every morphism in the source category and is described using the images in the target category under the action of both functors.

The identity functor and the single dual functor on finite-dimensional vector spaces are not naturally isomorphic. One obstruction is technical: the identity functor is covariant while the dual functor is contravariant. The essential failure of naturality is more significant. The isomorphisms between a vector space and its dual require a choice of basis, which is preserved by no nonidentity linear transformations.

abelian groups is determined by the image $g := \varphi(1_{\mathbb{Z}}) \in G$. Thus, the commutativity of the diagram

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{\alpha_{\mathbb{Z}}} & \mathbb{Z} \\ g \downarrow & & \downarrow g \\ G & \xrightarrow{\alpha_G} & G \end{array}$$

forces $\alpha_G(g) = n g$.

From the case $G = \mathbb{Z}$, it follows that the natural transformation defined by $G \rightarrow G/\tau(G) \rightarrow \tau(G) \oplus (G/\tau(G)) \cong G$ is multiplication by a nonzero integer n . Consider $G := \mathbb{Z}/\langle 2n \rangle$. This is a torsion group, so any map, such as $\alpha_{\mathbb{Z}/\langle 2n \rangle}$ which factors through the quotient, is zero. However, we have $n \neq 0 \in \mathbb{Z}/\langle 2n \rangle$ which is a contradiction. $\square$

Natural isomorphisms lead to the equivalence of categories.

12.1.5 Definition. An *equivalence of categories* consists of functors $F: \mathbf{C} \rightarrow \mathbf{D}$ and $G: \mathbf{D} \rightarrow \mathbf{C}$, and natural isomorphisms $\eta: \text{id}_{\mathbf{C}} \xrightarrow{\cong} G F$ and $\epsilon: \text{id}_{\mathbf{D}} \xrightarrow{\cong} F G$. The categories $\mathbf{C}$ and $\mathbf{D}$ are *equivalent*, written $\mathbf{C} \simeq \mathbf{D}$, if there exists an equivalence between them.

As the nomenclature suggests, the notion of equivalent categories defines an equivalence relation.

There is a useful characterization of the functors in an equivalence of categories. Its statement requires new terminology.

12.1.6 Definition. Let $F: \mathbf{C} \rightarrow \mathbf{D}$ be a functor. This functor is *full* if, for any objects X and Y in $\mathbf{C}$, the morphism map $\text{Hom}_{\mathbf{C}}(X, Y) \rightarrow \text{Hom}_{\mathbf{D}}(F(X), F(Y))$ is surjective. It is *faithful* if, for any objects X and Y in $\mathbf{C}$, the morphism map $\text{Hom}_{\mathbf{C}}(X, Y) \rightarrow \text{Hom}_{\mathbf{D}}(F(X), F(Y))$ is injective. Finally, it is *essentially surjective on objects* if, for every object Z in $\mathbf{D}$, there exists an object X in $\mathbf{C}$ such that Z is isomorphic to $F(X)$.

12.1.7 Theorem. A functor that defines an equivalence of categories is full, faithful, and essentially surjective on objects. Conversely, any functor with these properties defines an equivalence of categories. $\blacksquare$

The proof of Theorem 12.1.7 involves a lengthy diagram chase. The second part uses the axiom of choice.

12.1.8 Example. For any field K , the categories $\mathbf{Mat}_K$ and $\mathbf{Vect}_K^{\text{fd}}$ are equivalent. Consider an intermediate category $\mathbf{Vect}_K^{\text{bs}}$ whose objects are finite-dimensional vector spaces equipped with a chosen basis and whose morphisms are arbitrary linear maps. These categories are related by four functors: $U: \mathbf{Vect}_K^{\text{bs}} \rightarrow \mathbf{Vect}_K^{\text{fd}}$ is the forgetful functor. The functor $K^{(-)}: \mathbf{Mat}_K \rightarrow \mathbf{Vect}_K^{\text{bs}}$ sends n to the vector space K^n , equipped with the standard basis, and each $(m \times n)$ -matrix, relative to the standard bases on K^n and K^m , defines a linear map $K^n \rightarrow K^m$. The functor $H: \mathbf{Vect}_K^{\text{bs}} \rightarrow \mathbf{Mat}_K$ carries a vector space to its dimension and a linear map $\varphi: V \rightarrow W$ to the matrix relative to the chosen bases. The functor $C: \mathbf{Vect}_K^{\text{fd}} \rightarrow \mathbf{Vect}_K^{\text{bs}}$ is defined by choosing a basis for each vector space. One verifies that functors are full, faithful, and essentially surjective on objects. $\diamond$

12.2 Representable Functors

A foundational lemma in category theory demonstrates that every object can be characterized by a *universal property*.

12.2.1 Definition. Let F be a covariant or contravariant functor from a locally small category $\mathbf{C}$ to $\mathbf{Set}$. A *representation* of the functor F is a choice of object X in $\mathbf{C}$ with a natural isomorphism $\mathrm{Hom}_{\mathbf{C}}(X, -) \cong F$ when F is covariant, or $\mathrm{Hom}_{\mathbf{C}}(-, X) \cong F$ when F is contravariant. In either case, F is *represented by* X . A functor is *representable* if there exists a representation.

12.2.2 Example. The identity $\mathrm{id}_{\mathbf{Set}} : \mathbf{Set} \rightarrow \mathbf{Set}$ is represented by the singleton set $\{\emptyset\}$. For any set X , there exists a natural isomorphism $\mathrm{Hom}_{\mathbf{Set}}(\{\emptyset\}, X) \cong X$ that defines a bijection between the elements $x \in X$ and maps $x : \{\emptyset\} \rightarrow X$ carrying the singleton element to x . Naturality says that, for any $f : X \rightarrow Y$, the diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbf{Set}}(\{\emptyset\}, X) & \xrightarrow{\cong} & X \\ f_* \downarrow & & \downarrow f \\ \mathrm{Hom}_{\mathbf{Set}}(\{\emptyset\}, Y) & \xrightarrow{\cong} & Y \end{array}$$

commutes. The composite function $\{\emptyset\} \xrightarrow{x} X \xrightarrow{f} Y$ corresponds to the element $f(x)$. $\diamond$

12.2.3 Example. The forgetful functor $U : \mathbf{Top} \rightarrow \mathbf{Set}$ is represented by the singleton space. There exists a natural bijection between elements of a topological space and continuous functions from the one-point space. $\diamond$

12.2.4 Example. The forgetful functor $U : \mathbf{Grp} \rightarrow \mathbf{Set}$ is represented by the group $\mathbb{Z}$. For any group G , there is a natural isomorphism $\mathrm{Hom}_{\mathbf{Grp}}(\mathbb{Z}, G) \cong U(G)$ that associates, to each element g in $U(G)$, the unique group homomorphism $\mathbb{Z} \rightarrow G$ that maps $1_{\mathbb{Z}}$ to g . This defines a bijection because a group homomorphism $\mathbb{Z} \rightarrow G$ is determined by the image of $1_{\mathbb{Z}}$. In other words, $\mathbb{Z}$ is the free group on a single generator. This bijection is natural: for any morphism $\varphi : G \rightarrow H$ in $\mathbf{Grp}$, the composite $\mathbb{Z} \rightarrow G \rightarrow H$ sends $1_{\mathbb{Z}}$ to $\varphi(g)$. $\diamond$

12.2.5 Example. For any commutative ring R , the forgetful functor $U : \mathbf{Mod}_R \rightarrow \mathbf{Set}$ is represented by the R -module R^1 . There exists a natural bijection between R -linear maps $R \rightarrow V$ and the elements of the underlying set of V : an element v in $U(V)$ corresponds to the unique R -linear map that sends 1_R to v . In other words, R^1 is the free R -module on a single generator. $\diamond$

12.2.6 Example. The functor $U : \mathbf{CRng} \rightarrow \mathbf{Set}$ is represented by the ring $\mathbb{Z}[x]$. A ring homomorphism $\mathbb{Z}[x] \rightarrow R$ is uniquely determined

Certain classes of universal properties define blueprints which specify how a new object may be build out of a collection of existing ones.

A universal property of an object X in the locally small category $\mathbf{C}$ describes either the covariant functor $\mathrm{Hom}_{\mathbf{C}}(X, -)$ or the contravariant functor $\mathrm{Hom}_{\mathbf{C}}(-, X)$.

by the image of x . In other words, $\mathbb{Z}[x]$ is the free commutative ring on a single generator. $\diamond$

12.2.7 Example. Let n be a nonnegative integer and consider the following analogous functors. The functor $U^n : \mathbf{Grp} \rightarrow \mathbf{Set}$, which sends a group G to the set of n -tuples of elements of G , is represented by the free group on n generators. For any commutative ring R , the functor $U^n : \mathbf{Mod}_R \rightarrow \mathbf{Set}$ is represented by the free R -module R^n . Lastly, the functor $U^n : \mathbf{CRng} \rightarrow \mathbf{Set}$ is represented by the polynomial algebra $\mathbb{Z}[x_1, x_2, \dots, x_n]$. $\diamond$

The adjective “free” is reserved for universal properties expressed by covariant represented functors.

12.2.8 Example. The functor $(-)^{\times} : \mathbf{CRng} \rightarrow \mathbf{Set}$ that sends a ring to its units is represented by the Laurent polynomial ring $\mathbb{Z}[x, x^{-1}]$. A ring homomorphism $\mathbb{Z}[x, x^{-1}] \rightarrow R$ may be defined by sending x to any unit of R and is completely determined by this assignment. No ring homomorphism sends x to a nonunit. $\diamond$

12.2.9 Example. The contravariant power set functor $P : \mathbf{Set}^{\text{op}} \rightarrow \mathbf{Set}$ is represented by the set $\Omega := \{0, 1\}$ with two elements. The natural isomorphism $\text{Hom}_{\mathbf{Set}}(X, \Omega) \cong P(X)$ is defined by the bijection that associates a map $X \rightarrow \Omega$ with the subset that is the preimage of 1. Reversing perspectives, a subset $X' \subseteq X$ is identified with its indicator function $\chi_{X'} : X \rightarrow \Omega$ which sends exactly the elements of X' to 1. The naturality condition stipulates that, for any map $f : X \rightarrow Y$, the diagram

$$\begin{array}{ccc} \text{Hom}_{\mathbf{Set}}(Y, \Omega) & \xrightarrow{\cong} & P(X) \\ f^* \downarrow & & \downarrow f^{-1} \\ \text{Hom}_{\mathbf{Set}}(X, \Omega) & \xrightarrow{\cong} & P(Y) \end{array}$$

commutes. Given an indicator function $\chi_{Y'} : Y \rightarrow \Omega$, the composite function $X \xrightarrow{f} Y \xrightarrow{\chi_{Y'}} \Omega$ determines the subset $f^{-1}(Y') \subset X$. $\diamond$

12.2.10 Example. For any field K , the functor $U^* : \mathbf{Vect}_K^{\text{op}} \rightarrow \mathbf{Vect}_K$ that sends a vector space to the set of vectors in its dual space is represented by the vector space K ; linear functionals $V \rightarrow K$ are, by definition, precisely the vectors in the dual space V^* . $\diamond$

12.2.11 Example. For any two sets Y and Z , the functor

$$\text{Hom}(- \times Y, Z) : \mathbf{Set}^{\text{op}} \rightarrow \mathbf{Set}$$

that sends a set X to the set of functions $X \times Y \rightarrow Z$ is represented by the set Z^Y of functions from Y to Z . Hence, there exists a natural bijection between functions $X \times Y \rightarrow Z$ and functions $X \rightarrow Z^Y$. This natural isomorphism is referred to as *currying* in computer science. By fixing a variable in a two-variable function, one obtains a family of functions in a single variable. $\diamond$

12.3 The Yoneda Lemma

The previous section suggests that a representation encodes some sort of universal property of its representing object. If two objects represent the same functor, are they isomorphic?

This eponymous lemma was baptized by [Saunders Mac Lane](#) after learning about it from [Nobuo Yoneda](#) in 1954.

12.3.1 Definition. For any categories $\mathbf{C}$ and $\mathbf{D}$, the *functor category* $\mathbf{D}^{\mathbf{C}}$ has the functors $F: \mathbf{C} \rightarrow \mathbf{D}$ as the objects and natural transformations between them as morphisms. Given an object F in $\mathbf{D}^{\mathbf{C}}$, its identity morphism $\text{id}_F: F \Rightarrow F$ in $\mathbf{D}^{\mathbf{C}}$ is the natural transformation determined by $(\text{id}_F)_X := \text{id}_{F(X)}$ for all objects X in $\mathbf{C}$. To describe composition in $\mathbf{D}^{\mathbf{C}}$, consider three parallel functors $E, F, G: \mathbf{C} \rightarrow \mathbf{D}$ and two natural transformations $\alpha: E \Rightarrow F$ and $\beta: F \Rightarrow G$. The composite natural transformation $\beta \alpha: E \Rightarrow G$ is determined, for all objects X in $\mathbf{C}$, by $(\beta \alpha)_X := \beta_X \alpha_X$. Naturality of α and β implies that, for any morphism $f: X \rightarrow Y$ in $\mathbf{C}$, each square in the diagram

$$\begin{array}{ccccc} E(X) & \xrightarrow{\alpha_X} & F(X) & \xrightarrow{\beta_X} & G(X) \\ E(f) \downarrow & & F(f) \downarrow & & \downarrow G(f) \\ E(Y) & \xrightarrow{\alpha_Y} & F(Y) & \xrightarrow{\beta_Y} & G(Y) \end{array}$$

commutes, so the composite rectangle also commutes. To remain to verify that composition is associative and unital. It suffices to verify these properties for all objects X in $\mathbf{C}$. Therefore, they follow from the associativity and unitality of composition in $\mathbf{D}$.

The next result is arguably the most important result in category theory. It takes time to appreciate this deceptively deep statement.

12.3.2 Yoneda Lemma. *Let $\mathbf{C}$ be a locally small category. For any functor $F: \mathbf{C} \rightarrow \mathbf{Set}$ and any object X in $\mathbf{C}$, there is a bijection*

$$\text{Hom}_{\mathbf{Set}^{\mathbf{C}}}(\text{Hom}_{\mathbf{C}}(X, -), F) \cong F(X)$$

that associates a natural transformation $\alpha: \text{Hom}_{\mathbf{C}}(X, -) \Rightarrow F$ to the element $\alpha_X(\text{id}_X)$ in $F(X)$. This bijection is natural in both X and F . ■

The statement of the dual form of the Yoneda Lemma is left as an exercise.

A particular case of the Yoneda lemma characterizes the natural transformations between representable functors. Each object X in the category $\mathbf{C}$ represents a functor $\text{Hom}_{\mathbf{C}}(X, -): \mathbf{C} \rightarrow \mathbf{Set}$ and each morphism $f: X \rightarrow Y$ in $\mathbf{C}$ corresponds to a natural transformation $f^*: \text{Hom}_{\mathbf{C}}(Y, -) \Rightarrow \text{Hom}_{\mathbf{C}}(X, -)$ determined, for all objects Z in $\mathbf{C}$, by the precomposition map $f^*: \text{Hom}_{\mathbf{C}}(Y, Z) \rightarrow \text{Hom}_{\mathbf{C}}(X, Z)$. This data determines a functor $H: \mathbf{C}^{\text{op}} \rightarrow \mathbf{Set}^{\mathbf{C}}$.

12.3.3 Corollary. *The functor $H: \mathbf{C}^{\text{op}} \rightarrow \mathbf{Set}^{\mathbf{C}}$ is full and faithful.*

The functor H is called the *Yoneda embedding*.

Proof. By Definition 12.1.6, the functor $H: \mathbf{C}^{\text{op}} \rightarrow \mathbf{Set}^{\mathbf{C}}$ is full and faithful provided it defines local bijections between hom-sets

$$\mathrm{Hom}_{\mathbf{C}}(X, Y) \cong \mathrm{Hom}_{\mathrm{Set}^{\mathbf{C}}}(\mathrm{Hom}_{\mathbf{C}}(Y, -), \mathrm{Hom}_{\mathbf{C}}(X, -)).$$

The definition of the functor H ensures that this map is injective: distinct morphism induces distinct natural transformations. The Yoneda lemma implies that a natural transformation

$$\alpha: \mathrm{Hom}_{\mathbf{C}}(Y, -) \Rightarrow \mathrm{Hom}_{\mathbf{C}}(X, -)$$

corresponds to elements in $\mathrm{Hom}_{\mathbf{C}}(X, Y)$; the morphism $f: X \rightarrow Y$ in $\mathbf{C}$ is $f = \alpha_Y(\mathrm{id}_Y)$. Defined as precomposition with $f: X \rightarrow Y$, the natural transformation $f^*: \mathrm{Hom}_{\mathbf{C}}(Y, -) \Rightarrow \mathrm{Hom}_{\mathbf{C}}(X, -)$ sends id_Y to f . Thus, the bijection implies that $\alpha = f^*$. $\square$

12.3.4 Corollary. *Every group is isomorphic to a subgroup of a symmetric group.*

Proof. For any group G , Example 12.0.10 establishes that a functor $X: \mathbf{BG} \rightarrow \mathbf{Set}$ corresponds to a set X with a G -action. A natural transformation $\alpha: X \Rightarrow Y$ consists of a single G -equivariant map $\alpha: Y \rightarrow X$ of sets. The G -equivariant maps $G \rightarrow X$ are in bijection with elements of X : identify a map with the image of e in G . Hence, the image of the Yoneda embedding $\mathbf{BG}^{\mathrm{op}} \rightarrow \mathbf{Set}^{\mathbf{BG}}$ is the set G under left translation. Corollary 12.3.3 implies that the only G -equivariant endomorphisms of G are those defined by right multiplication with an element of G . In particular, any G -equivariant endomorphism of G must be an automorphism.

Thus, the Yoneda embedding defines an isomorphism between G and the automorphism group of the G , regarded as an object in $\mathbf{Set}^{\mathbf{BG}}$. Composing with the faithful forgetful functor $\mathbf{Set}^{\mathbf{BG}} \rightarrow \mathbf{Set}$ gives an isomorphism between G and a subgroup of the automorphism group $\mathfrak{S}_G$ of the set G . $\square$

12.3.5 Corollary. *Let X and Y be objects in a locally small category $\mathbf{C}$. When the functors represented by X and Y are naturally isomorphic, the objects X and Y are isomorphic. In particular, when X and Y represent the same functor, the objects X and Y are isomorphic.*

Sketch of proof. The full and faithful functor $H: \mathbf{C}^{\mathrm{op}} \rightarrow \mathbf{Set}^{\mathbf{C}}$ creates isomorphisms: any two objects in $\mathbf{C}^{\mathrm{op}}$, whose images are isomorphic in $\mathbf{Set}^{\mathbf{C}}$, are isomorphic in $\mathbf{C}^{\mathrm{op}}$. Thus, an isomorphism between represented functors is induced by a unique isomorphism between their representing objects. Given a functor represented by X and Y , the representing natural isomorphisms compose to show that X and Y are representably isomorphic. $\square$

The Yoneda lemma is a generalization of Theorem 2.1.8.

A map $\alpha: Y \rightarrow X$ is G -equivariant if, for any g in G , the diagram

$$\begin{array}{ccc} Y & \xrightarrow{\alpha} & X \\ g^* \downarrow & & \downarrow g^* \\ Y & \xrightarrow{\alpha} & X \end{array}$$

commutes.

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