

Problems 02

Due: Thursday, 18 September 2025 before 23:59 ET

1. Let X and Y be sets. The identity map $\text{id}_X : X \rightarrow X$ is defined by $x \mapsto x$ for all x in X , and the projection $\pi_1 : X \times Y \rightarrow X$ is defined by $(x, y) \mapsto x$ for all x in X and all y in Y . For any maps $\varphi : X \rightarrow X$ and $\psi : X \rightarrow Y$, the map $\varphi \parallel \psi : X \rightarrow X \times Y$ is defined by $x \mapsto (\varphi(x), \psi(x))$ for all x in X , and the map $\varphi \times \psi : X \times X \rightarrow X \times Y$ is defined by $(w, x) \mapsto (\varphi(w), \psi(x))$ for all w and all x in X . For the 1-element set $\{\emptyset\}$, there exists a unique map $\eta : X \rightarrow \{\emptyset\}$ defined by $\eta(x) = \emptyset$ for all x in X .

Suppose that the set X is nonempty and consider three maps $\beta : X \times X \rightarrow X$, $\varepsilon : \{\emptyset\} \rightarrow X$, and $\iota : X \rightarrow X$ satisfying the following three conditions:

- (associativity) $\beta \circ (\beta \times \text{id}_X) = \beta \circ (\text{id}_X \times \beta)$,
(right identity) $\beta \circ (\text{id}_X \times \varepsilon) = \text{id}_X \circ \pi_1$,
(right inverse) $\beta \circ (\text{id}_X \parallel \iota) = \varepsilon \circ \eta$.

Prove that the quadruple $(X, \beta, \varepsilon, \iota)$ defines a group.

2. For any nonnegative integer n , the **sign function** $\text{sgn} : \mathfrak{S}_n \rightarrow \mu_2 := \{\pm 1\}$ is defined by $\text{sgn}(\sigma) := (-1)^{n-c}$ where the permutation σ is the product of c disjoint cycles.
- For any permutation σ and any transposition ϖ , prove $\text{sgn}(\varpi \sigma) = -\text{sgn}(\sigma)$.
 - For any permutations σ and τ , show that $\text{sgn}(\sigma \tau) = \text{sgn}(\sigma) \text{sgn}(\tau)$.
 - When the permutation σ is the product of m transpositions, demonstrate that $\text{sgn}(\sigma) = (-1)^m$.

3. The **quaternion group** is the subgroup of $\text{SL}(2, \mathbb{C})$ generated by the eight matrices:

$$\begin{aligned} \mathbf{I} &:= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, & \mathbf{A} &:= \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, & \mathbf{B} &:= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, & \mathbf{C} &:= \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \\ -\mathbf{I} &:= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, & -\mathbf{A} &:= \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}, & -\mathbf{B} &:= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, & -\mathbf{C} &:= \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}. \end{aligned}$$

- Determine the order of the quaternion group.
- Find a minimal set of generators for the quaternion group.
- Show that the quaternion group is not isomorphic to the dihedral group D_4 .