

Problems 03

Due: Thursday, 25 September 2025 before 23:59 ET

1. Let H and K be subgroups of a group G . For any element g in G , the set

$$H g K := \{f \in G \mid f = h g k \text{ for some } h \in H, k \in K\}$$

is called a **double coset**.

- i. Prove that the double cosets partition G .
 - ii. Do all double cosets have the same cardinality?
 - iii. When G has finite order, must the cardinality of a double coset divide $|G|$?
2. Let G be a group and let $\text{Aut}(G)$ be its automorphism group. For any element g in G , consider the map $\gamma_g: G \rightarrow G$ defined by $\gamma_g(f) := g f g^{-1}$ for all f in G .
- i. For any element g in G , show that γ_g is an automorphism.
 - ii. Prove that the map $\Gamma: G \rightarrow \text{Aut}(G)$ defined by $\Gamma(g) := \gamma_g$, for all g in G , is a group homomorphism.
 - iii. Show that $\text{Ker}(\Gamma) = Z(G)$.
 - iv. Prove that the image $\text{Im}(\Gamma)$ is a normal subgroup of $\text{Aut}(G)$.
3. Fix a nonnegative integer n . The permutations σ and τ in the symmetric group $\mathfrak{S}_n$ have the **same cycle structure** if, for any nonnegative integer k , their factorizations into disjoint cycles have the same number of cycles of length k . The **cycle type** of a permutation is the list $\lambda := (\lambda_n, \lambda_{n-1}, \dots, \lambda_1)$ where λ_j is the number of cycles length j in its factorization into disjoint cycles.
- i. For any permutations σ and τ in $\mathfrak{S}_n$, prove that the conjugate permutation $\sigma \tau \sigma^{-1}$ has the same cycle structure as τ and may be obtained by applying σ to the entries in the cycles of τ .
 - ii. Prove that permutations are conjugate if and only if they have the same cycle type.