

Problems 04

Due: Thursday, 2 October 2025 before 23:59 ET

1. Let G be a group. The **commutator** of the elements f and g in G is the element $[f, g] := f^{-1}g^{-1}fg$ in G . The **commutator subgroup** $G^{(1)}$ of G is the subgroup generated by all commutators; $G^{(1)} := \langle f^{-1}g^{-1}fg \mid f, g \in G \rangle$.
 - i. Prove that $G^{(1)}$ is a normal subgroup and the quotient group $G/G^{(1)}$ is abelian.
 - ii. Let $\pi: G \rightarrow G/G^{(1)}$ be the canonical group homomorphism. For any abelian group A , demonstrate that every group homomorphism $\varphi: G \rightarrow A$ factors as $\varphi = \varphi^{(1)} \circ \pi$ where $\varphi^{(1)}: G/G^{(1)} \rightarrow A/A^{(1)}$ is the induced group homomorphism.
 - iii. Show that a subgroup H of G contains $G^{(1)}$ if and only if H is normal and G/H is abelian.
2. Let G be a group, let H be a subgroup of G , and let $N_G(H) := \{g \in G \mid gHg^{-1} = H\}$ be the normalizer of H .
 - i. Show that $N_G(H)$ is a subgroup of G .
 - ii. Show $N_G(H)$ acts on the set G/H of all left cosets for the subgroup H via the map $N_G(H) \times G/H \rightarrow G/H$ defined by $f(gH) := gHf^{-1} = gf^{-1}H$ for any element f in $N_G(H)$ and any element g in G .
 - iii. Show that the induced map $\hat{\varphi}: N_G(H)/H \rightarrow \mathfrak{S}_{G/H}$ is injective.
3. The **icosahedral group** I is the group of the rotational symmetries of a regular dodecahedron (or regular icosahedron). It acts transitively on the vertices, edges, and faces. Moreover, we have $|I| = 60$.
 - i. Determine the number of elements in I of each order.
 - ii. Determine the cardinality of each conjugacy class in I .
 - iii. Show that I is a simple group (i.e. it has no nontrivial normal subgroups).