

Problems 07

Due: Thursday, 30 October 2025 before 23:59 ET

1. For any field K , let $K((x))$ be ring of formal Laurent series with coefficients in K :

$$K((x)) := \left\{ \sum_{i=m}^{\infty} a_i x^i \mid a_i \in K, m \in \mathbb{Z} \right\}$$

where the ring operations are defined as in the ring of formal power series $K[[x]]$. Prove that the commutative ring $K((x))$ is isomorphic to the fields of fractions for the domain $K[[x]]$.

2. Let R be a commutative ring. Consider the **derivative operator** $D : R[x] \rightarrow R[x]$ defined, for any polynomial $f := a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ in $R[x]$, by

$$D(f) := (n a_n) x^{n-1} + ((n-1) a_{n-1}) x^{n-2} + \cdots + a_1.$$

- i. Prove that the operator D is an R -linear map: for any r and s in the coefficient ring R , and any f and g in the polynomial ring $R[x]$, we have

$$D(r f + s g) = r D(f) + s D(g).$$

- ii. Prove that the operator D satisfies the Leibniz product rule: for any f and g in the polynomial ring $R[x]$, we have

$$D(f g) = D(f) g + f D(g).$$

- iii. Let f be a polynomial in $R[x]$. Assume that z in R is root of f having multiplicity m with $m \geq 1$. Prove that z is also a root of the derivative $D(f)$ having multiplicity at least $m - 1$. Moreover, when the product $m 1_R$ is invertible in R , prove that z is a root of the derivative $D(f)$ having multiplicity $m - 1$.

3. Let $\omega := \frac{1}{2}(-1 + \sqrt{3}i)$ in $\mathbb{C}$ be one of the complex roots of the polynomial $x^2 + x + 1$ in $\mathbb{C}[x]$. Prove that the commutative domain $\mathbb{Z}[\omega] := \{a + b \omega \mid a, b \in \mathbb{Z}\} \subset \mathbb{C}$ is a Euclidean domain with the function $\nu : \mathbb{Z}[\omega] \rightarrow \mathbb{N}$ defined, for any $a + b \omega$ in $\mathbb{Z}[\omega]$, by $\nu(a + b \omega) = a^2 - a b + b^2$.