

# Problems 08

Due: Thursday, 6 November 2025 before 23:59 ET

1. Euclid proves that there are infinitely many prime integers in the following way: if  $p_1, p_2, \dots, p_k$  are prime numbers, then any prime factor of the integer  $1 + p_1 p_2 \cdots p_k$  must be different from  $p_i$  for all  $1 \leq i \leq k$ .
  - i. Adapt this argument to demonstrate that, for any field  $K$ , there are infinitely many monic irreducible polynomials in  $K[x]$ .
  - ii. Explain why the argument fails for the formal power series ring  $K[[x]]$  over a field  $K$ .
  - iii. Adapt this argument to show that the set of prime integers of the form  $4n - 1$  is infinite.

2. Let  $R$  be a principal ideal domain and let  $K$  be its field of fractions.
  - i. Suppose  $R = \mathbb{Z}$ . Write  $r = 11/108 \in \mathbb{Q}$  in the form  $r = a/27 + b/4$ .
  - ii. Consider  $g := pq$  in  $R$  where  $p$  and  $q$  are relatively prime. Prove that every fraction  $f/g \in K$  can be written in the form

$$\frac{f}{g} = \frac{a}{q} + \frac{b}{p}$$

for some  $a$  and  $b$  in  $R$ .

- iii. Let  $k$  be a positive integer and let  $g := p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k}$  be the factorization of the element  $g$  in  $R$  into irreducible elements  $p_1, p_2, \dots, p_k$  such that the relation  $p_i = u p_j$  for some unit  $u$  in  $R$  implies that  $i = j$ . Prove that every fraction  $r = f/g \in K$  can be written in the form

$$r = \sum_{i=1}^k \frac{h_i}{p_i^{m_i}}$$

for some  $h_i$  in  $R$  for all  $1 \leq i \leq k$ .

3. Let  $R$  be a unique factorization domain such that the sum of two principal ideals in  $R$  is again a principal ideal. Prove that  $R$  is a principal ideal domain.