

Problems 10

Due: Thursday, 20 November 2025 before 23:59 ET

1. A module is **simple** if it is not the zero module and if it has no proper submodule.
 - i. Let V be a simple R -module. Show that V is cyclic.
 - ii. Prove *Schur's Lemma*: Any R -linear map $\varphi : V \rightarrow W$ between simple R -modules is either zero or an isomorphism.
 - iii. For a simple R -module V , show that the set $\text{End}_R(V)$ of endomorphisms forms a field where multiplication is given by composition of functions and addition is defined pointwise.
2. Let R be a domain and let V be an R -module. An element v in V is a **torsion element** if there is a nonzero element r in R such that $rv = 0$. Let $\tau(V)$ be the set of torsion elements of V . A module V is **torsion** if $\tau(V) = V$ and it is **torsion-free** if $\tau(V) = 0$.
 - i. Demonstrate that the **annihilator** $\text{Ann}(V) := \{f \in R \mid f v = 0 \text{ for all } v \in V\}$ forms an ideal in R .
 - ii. Show that the set $\tau(V)$ is a submodule of V .
 - iii. Prove that the quotient R -module $V/\tau(V)$ is torsion-free.
 - iv. For any R -linear map $\varphi : V \rightarrow W$, demonstrate that $\varphi(\tau(V)) \subseteq \tau(W)$.
 - v. Give an example of an infinite abelian group that is a torsion $\mathbb{Z}$ -module.
3. i. Let $\varphi : V' \rightarrow V$ and $\psi : V \rightarrow V''$ be R -linear maps. Prove that the sequence
 - ($\ddagger$)
$$V' \xrightarrow{\varphi} V \xrightarrow{\psi} V'' \longrightarrow 0$$
is exact if and only if, for every R -module W , the sequence
 - ($\star$)
$$0 \longrightarrow \text{Hom}_R(V'', W) \xrightarrow{\text{Hom}_R(\psi, W)} \text{Hom}_R(V, W) \xrightarrow{\text{Hom}_R(\varphi, W)} \text{Hom}_R(V', W)$$
is exact.
 - ii. Show that $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/\langle m \rangle, \mathbb{Z}/\langle n \rangle) \cong \mathbb{Z}/\langle d \rangle$ where $d := \gcd(m, n)$.