

Problems 12

Due: Thursday, 4 December 2025 before 23:59 ET

1. Consider the integer matrix

$$\mathbf{B} := \begin{bmatrix} 1 & 3 & 4 & 2 \\ 5 & 15 & -4 & 34 \\ -3 & -3 & 36 & -24 \end{bmatrix}.$$

- i. Find a (3×3) -matrix $\mathbf{Q}$ over $\mathbb{Z}$ and a (4×4) -matrix $\mathbf{P}$ over $\mathbb{Z}$ such that $\mathbf{QBP}$ is in Smith normal form.
- ii. Let G be an abelian group with generators g_1, g_2, g_3 and g_4 , and the relations

$$\begin{aligned} 1g_1 + 3g_2 + 4g_3 + 2g_4 &= 0 \\ 5g_1 + 15g_2 - 4g_3 + 34g_4 &= 0 \\ -3g_1 - 3g_2 + 36g_3 - 24g_4 &= 0. \end{aligned}$$

Show that $G \cong \mathbb{Z}/\langle 6 \rangle \oplus \mathbb{Z}/\langle 24 \rangle \oplus \mathbb{Z}$. In addition, find new generators h_1, h_2 and h_3 such that $6h_1 = 0$, $24h_2 = 0$ and h_3 has infinite order.

- iii. Find all integer (3×1) -matrices $\mathbf{X}$ such that $\mathbf{BX} = \mathbf{C}$ where

$$\mathbf{C} := \begin{bmatrix} 7 \\ 11 \\ -9 \end{bmatrix}.$$

2. Consider the matrix

$$\mathbf{A} := \begin{bmatrix} 2 & 0 & -1 & -5 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & -1 & -1 \\ 1 & 0 & -1 & -4 \end{bmatrix}.$$

- i. Find the minimal polynomial of $\mathbf{A}$.
- ii. Find the rational canonical form of $\mathbf{A}$.
- iii. Find the Jordan canonical form of $\mathbf{A}$.

3. A **poset** is a set P with a binary relation $\leq_P$ that is reflexive, antisymmetric, and transitive. For any elements x, y , and z in P , we have the following:

(reflexivity) The relation $x \leq_P x$ holds.

(antisymmetry) The relations $x \leq_P y$ and $y \leq_P x$ imply that $x = y$.

(transitivity) The relations $x \leq_P y$ and $y \leq_P z$ imply that $x \leq_P z$.

A function $f: P \rightarrow Q$ between posets is **order-preserving** if, for any x and y in P satisfying $x \leq_P y$, we have $f(x) \leq_Q f(y)$.

- i. Show that the collection of posets together with order-preserving functions forms a category.
- ii. In the category of posets, exhibit a bijective morphism between nonisomorphic posets.